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Exact solution for the resistance of random walks on a Cayley tree

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Abstract. We consider random walks consisting of successive steps between neighbouring sites on a Cayley tree of arbitrary coordination number $z = \sigma + 1$. Each bond between neighbouring sites is assigned a conductance qg_0 , where q is the number of times the bond is traversed in the random walk. We present an exact evaluation of the resistive susceptibility from which we find the end-to-end resistance averaged over all z^N walks of length N is given by $[R]^{(N)} \sim N$ for $\sigma > 1$ and $[R]^{(N)} \sim \frac{1}{2} \ln N$ for $\sigma = 1$, i.e. for a one-dimensional chain.

1. Introduction

Motivated by problems concerning fluid flow in low porosity media, Banavar *et al* (1983) introduced a model whose analogue electrical network gives rise to some interesting statistical problems. In particular, we give here the exact solution on a Cayley tree of arbitrary coordination number z to the following problem. Consider the ensemble $\Gamma^{(N)}$ of z^N random walks, each of which consists of N successive steps between neighbouring sites on a Cayley tree (a section of which is shown for $z = 3$ in figure 1). Let $R(\gamma)$ be the end-to-end resistance of the walk γ obtained if we associate with each step a unit conductance g_0 . If a bond between neighbouring sites is traversed

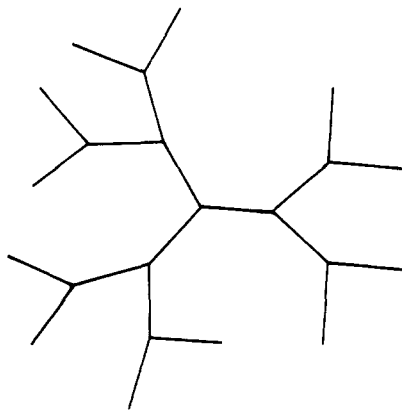


Figure 1. Section of a Cayley tree with coordination number $\sigma + 1 = 3$. For $\sigma = 1$, the Cayley tree is a one-dimensional chain.

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q times in the random walk, then it is assigned a conductance qg_0 . Some examples of random walks with their resistances are shown in figure 2. The object of this study will be the calculation of the 'resistive susceptibility', $\chi(K)$, defined by

$$\chi(K) = \sum_N \sum_{\gamma \in \Gamma^{(N)}} R(\gamma) K^N \tag{1}$$

where K is a fugacity, and the sum over N and $\Gamma^{(N)}$ is equivalent to summing over all walks. The notation $\gamma \in \Gamma^{(N)}$ indicates that the sum is over walks γ in the ensemble $\Gamma^{(N)}$ of N -step walks.

For fixed starting point, the distribution of endpoints of the ensemble of random walks is known (Barber and Ninham 1970) to be Gaussian for large N in all spatial dimensions d . The resistive susceptibility diverges at $K = z^{-1}$ and arguments (Harris and Christou 1987) based on the Gaussian nature of random walk intersections or an analysis of the associated field theory from which χ may be determined both show that, for $d > 4$,

$$\chi \sim C(z^{-1} - K)^{-2} \quad K \rightarrow z^{-1} \tag{2}$$

where C is an indefinite constant. This implies that

$$[R]^{(N)} \sim CN \tag{3}$$

where $[R]^{(N)}$ denotes the average of $R(\gamma)$ over the ensemble of walks of length N . For $d < 4$ one has

$$\chi \sim C(z^{-1} - K)^{-1-\phi} \tag{4}$$

and correspondingly

$$[R]^{(N)} \sim CN^\varphi \tag{5}$$

where φ is a crossover exponent. For d near 4 an expansion in $\varepsilon = 4 - d$ gives (Harris and Cristou 1987)

$$\varphi = 1 - \frac{1}{4}\varepsilon - \frac{1}{16}\varepsilon^2. \tag{6}$$

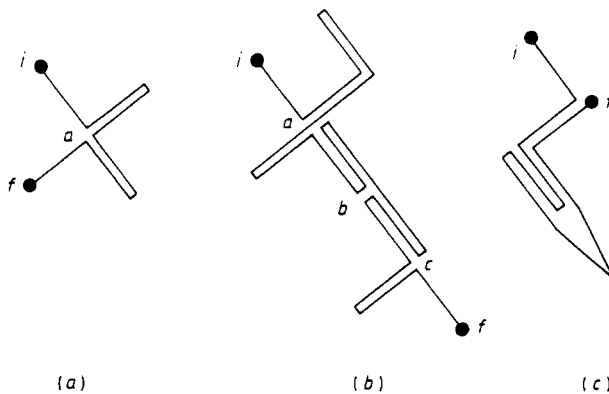


Figure 2. Examples of random walks of N steps from an initial site i to a final site f having end-to-end resistance R as indicated. (a) $N = 6$, $R = 2$ and the sum over (kl) in (12) is over (i, a) and (a, f) and the corresponding $n_{kl}(\gamma)$ are 1,1. (b) $N = 16$, $R = \frac{8}{3}$ and the sum is over (i, a) , (a, b) , (b, c) and (c, f) , and the corresponding $n_{kl}(\gamma)$ are 1,3,3,1. (c) $N = 9$, $R = 1$ and the sum has a single term (i, f) with a corresponding $n_{kl}(\gamma)$ of 1.

For $d = 1$ a hand-waving argument (Harris and Cristou 1987) suggested that $\varphi = 0$, which we will verify, in that we find for this case

$$\chi \sim C(z^{-1} - K) \ln(z^{-1} - K). \tag{7}$$

For a Cayley tree with coordination number greater than two, we recover (2) with no logarithmic corrections, as expected. The numerical data (Banavar *et al* 1983) are consistent with these results.

Briefly this paper is organised as follows. In § 2 we present the calculation and in § 3 we discuss our results and summarise the conclusions to be drawn therefrom.

2. Method of calculation

In this section we describe the exact calculation for χ which we have carried out for the Cayley tree. Unfortunately the method depends on the lattice not having parallel paths between two different sites. Accordingly, the method cannot be generalised to actual d -dimensional lattices with $1 < d < \infty$. Nevertheless, the answers are useful in that they represent the exact solutions for $d = 1$ and $d = \infty$. Briefly, the method we use is based upon evaluating a lattice Green function, and is therefore similar in spirit to that used by Montroll and Weiss (1965). However, we need to extend their method to count the number of times a bond is traversed, since this information enables us to obtain the resistance of a random walk.

Consider the Green function G_{ij} which we interpret as a matrix in the site indices i and j defined by

$$G_{ij}^{(0)} = [(E - H)^{-1}]_{ij} \tag{8}$$

where we set $E = 1$ and take H to be the hopping Hamiltonian

$$H_{ij} = K_{ij} \{ |i\rangle\langle j| + |j\rangle\langle i| \} \tag{9}$$

in bra-ket notation, where $K_{ij} = 0$ unless sites i and j are nearest neighbours. Normally, we would set $K_{ij} = K$ for all nearest-neighbouring interactions, in which case $G_{ij}^{(0)}$ is the random-walk generating function defined by

$$G_{ij}^{(0)} = \sum_N K^N \sum_{\gamma \in \Gamma_{ij}^{(N)}} 1 \tag{10}$$

where $\gamma \in \Gamma_{ij}^{(N)}$ indicates that the sum is over walks γ in the ensemble $\Gamma_{ij}^{(N)}$ of N -step walks which begin at site i and end at site j .

We introduce a perturbation which will allow us to count bond traversals. Suppose we modify one of the K (specifically K_{kl}) by setting it equal to K' . Then we write the corresponding Green function as

$$G_{ij}^{(kl)}(K, K') = \sum_N K^N \sum_{\gamma \in \Gamma_{ij}^{(N)}} (K'/K)^{n_{kl}(\gamma)} \tag{11}$$

where $n_{kl}(\gamma)$ is the number of times the bond connecting sites k and l is traversed in the walk γ . In one dimension or on a Cayley tree, in either case where there are no parallel paths, the end-to-end resistance of a walk between sites i and j is easily calculated in terms of the n_{kl}

$$R(\gamma) = \sum_{(kl) \in \gamma_{SAW}} (n_{kl}(\gamma))^{-1} \tag{12}$$

where the factor $(n_{kl}(\gamma))^{-1}$ takes account of the fact that the conductance of a bond which is traversed q times is taken to be qg_0 , where g_0 is a unit conductance which will be omitted henceforth. In the sum over bonds (kl) , we obviously only sum over the bonds which occur in the self-avoiding walk (SAW) connecting sites i and j (see figure 2) and this is indicated by the notation $(kl) \in \gamma_{SAW}$. Bonds not in this self-avoiding walk have no influence on the end-to-end resistance of the walk γ .

We can generate (12) by a suitable integration over K' in (11). If we define χ_{ij} to be the resistive susceptibility associated with random walks which begin at site i and end at site j , then we have

$$\chi_{ij} = \sum_{(kl) \in \gamma_{SAW}} \int_0^K G_{ij}^{(kl)}(K, K') \frac{dK'}{K'} \tag{13}$$

We can, however, include the perturbation in the hopping matrix element, $(K' - K)$ via an exact calculation (Koster and Slater 1954) of the t matrix for the perturbing bond (kl) . That is, we take

$$V^{(kl)} = (K' - K)\{|k\rangle\langle l| + |l\rangle\langle k|\} \tag{14}$$

and construct

$$t^{(kl)}(K') = V^{(kl)}[1 - G^{(0)}V^{(kl)}]^{-1} \tag{15}$$

in terms of which we have

$$G_{ij}^{(kl)}(K, K') = G_{ij}^{(0)} + \sum_{mn} G_{im}^{(0)} [t^{(kl)}(K')]_{mn} G_{nj}^{(0)} \tag{16}$$

Therefore we may write (13) in the form

$$\chi_{ij} = \sum_{(kl) \in \gamma_{SAW}} \int_0^K \frac{dK'}{K'} \left(G_{ij}^{(0)} + \sum_{mn} G_{im}^{(0)} [t^{(kl)}(K')]_{mn} G_{nj}^{(0)} \right) \tag{17}$$

Several simplifying features are noteworthy. First of all, the t matrix is local to the bond (kl) , so that in (17) the indices m and n range only over the values k and l . Also, the t matrix has the same form for all bonds (kl) . Note that if we were to integrate the term in (17) involving $G_{ij}^{(0)}$ separately, we would encounter a divergence at $K' = 0$. However, $G_{ij}^{(kl)}(K, K')$ vanishes for $K' = 0$ because when the hopping matrix element associated with bond (kl) is zero then sites i and j are decoupled. Therefore there is no divergence in (17) when the terms in $G^{(0)}$ and $G^{(0)}t^{(0)}$ are combined. This reasoning indicates that

$$G_{ij}^{(kl)}(K, 0) = 0 = G_{ij}^{(0)} + \sum_{mn} G_{im}^{(0)} G_{nj}^{(0)} t_{mn}^{(kl)}(0) \tag{18}$$

and this relation is used to write $G_{ij}^{(0)}$ in terms of the t matrix evaluated at $K' = 0$. Thus we have

$$\chi_{ij} = \sum_{(kl) \in \gamma_{SAW}} \sum_{mn} G_{im}^{(0)} G_{nj}^{(0)} \int_0^K \frac{dK'}{K'} [\Delta t^{(kl)}(K')]_{mn} \tag{19}$$

where

$$\Delta t(K') = t(K') - t(0). \tag{20}$$

In matrix notation, where the rows and columns correspond to the site labels k and l , we write (15) as

$$t^{(kl)}(K') = \begin{bmatrix} 0 & K' - K \\ K' - K & 0 \end{bmatrix} \begin{bmatrix} 1 - (K' - K)G_0 & -(K' - K)G_d \\ -(K' - K)G_d & 1 - (K' - K)G_0 \end{bmatrix}^{-1} \tag{21a}$$

where $G_d = G_{kk}^{(0)}$ is the on-site Green function and $G_0 = G_{kl}^{(0)}$ is the nearest-neighbour off-diagonal Green function. After some algebra we find that

$$\frac{t^{(kl)}(K') - t(kl)(0)}{K'} = \frac{1}{2} \begin{bmatrix} \alpha_- & \alpha_+ \\ \alpha_+ & \alpha_- \end{bmatrix} \tag{21b}$$

where

$$\alpha_{\pm} = [(1 + KG_+)(1 + (K - K')G_+)]^{-1} \pm [(1 + KG_-)(1 + (K - K')G_-)]^{-1} \tag{22}$$

and

$$G_{\pm} \equiv G_0 \pm G_d. \tag{23}$$

The integration over the coupling constant is now easily done:

$$\int_0^K \alpha_{\pm} dK' = \frac{\ln(1 + KG_+)}{G_+(1 + KG_+)} \pm \frac{\ln(1 + KG_-)}{G_-(1 + KG_-)}. \tag{24}$$

Then (19) takes the very simple form

$$\begin{aligned} \chi_{ij} = & \sum_{k \in \text{SAW}} \frac{1}{2} (G_{ik}^{(0)} + G_{i,k+1}^{(0)}) (G_{kj}^{(0)} + G_{k+1,j}^{(0)}) \frac{\ln(1 + KG_+)}{G_+(1 + KG_+)} \\ & - \sum_{k \in \text{SAW}} \frac{1}{2} (G_{ik}^{(0)} - G_{i,k+1}^{(0)}) (G_{kj}^{(0)} - G_{k+1,j}^{(0)}) \frac{\ln(1 + KG_-)}{G_-(1 + KG_-)}. \end{aligned} \tag{25}$$

The notation $k + 1$ means the site next to k but further in the direction of j . Here the sum over k is over all points between i and j , including i but not j . The restriction $k \neq j$ occurs because we have inserted the t matrix $t^{(k,k+1)}$. As we shall see, the sum over k will be trivial.

To evaluate (25) we need $G_{ij}^{(0)}$ for the homogeneous hopping model on the Cayley tree. For the Cayley tree note that $G_{ij}^{(0)}$ only depends on the number of steps n between sites i and j . Following the method of Harris and Lubensky (1981, appendix A) we have

$$G_{ij}^{(0)} = G_{ii}^{(0)} x_+^{-n_{ij}} = G_{ii}^{(0)} (x_- / \sigma)^{n_{ij}} \tag{26}$$

where n_{ij} is the number of steps between sites i and j . Here we have introduced

$$x_{\pm} = [1 \pm (1 - 4\sigma K^2)^{1/2}] / (2K) \tag{27}$$

where $\sigma = z - 1$, where z is the coordination number of the tree. Also $G_{ii}^{(0)}$ is given by

$$[G_{ii}^{(0)}]^{-1} = 1 - K \frac{(\sigma + 1)}{\sigma} x_-. \tag{28}$$

In this notation

$$G_{\pm} = G_{ii}^{(0)} \left(\frac{x_-}{\sigma} \pm 1 \right) \tag{29a}$$

$$1 + KG_{\pm} = KG_{ii}^{(0)} \left(\frac{\sigma}{x_-} \pm 1 \right). \tag{29b}$$

Due to the form of (26) one sees that the summand in (25) is actually independent of k . The number of allowed values of k is simply n_{ij} . Using this fact and (26) we write (25) as

$$\chi_{ij} = n_{ij} [G_{ii}^{(0)}]^2 \left(\frac{x_-}{\sigma}\right)^{n_{ij}} \left[\left(\frac{x_-}{\sigma} + \frac{\sigma}{x_-} + 2\right) \frac{\ln(1 + KG_+)}{2G_+(1 + KG_+)} + \left(\frac{x_-}{\sigma} + \frac{\sigma}{x_-} - 2\right) \frac{\ln(1 + KG_-)}{2G_-(1 + KG_-)} \right]. \tag{30a}$$

Using (29) we simplify (30a) into the form

$$\chi_{ij} = \frac{n_{ij}}{2K} \left(\frac{x_-}{\sigma}\right)^{n_{ij}} \ln\left(\frac{x_+ + 1}{x_+ - 1}\right). \tag{30b}$$

To get the total susceptibility we sum over shells indexed by n_{ij} . The number of sites in the shell n_{ij} is simply $(\sigma + 1)\sigma^{n_{ij}-1}$, for $n_{ij} \geq 1$. The final result is

$$\chi = \sum_{n_{ij}} (\sigma + 1)\sigma^{n_{ij}-1} \chi_{ij} = \frac{(\sigma + 1)x_-}{2K\sigma(1 - x_-)^2} \ln\left(\frac{x_+ + 1}{x_+ - 1}\right). \tag{31}$$

For $\sigma = 1$ we have the special case of a one-dimensional chain

$$\chi = \frac{1}{2(1 - 2K)} \ln\left(\frac{1 + 2K}{1 - 2K}\right). \tag{32}$$

The critical exponents are associated with the singularities at $K = z^{-1}$. For this value of K , $x_- \rightarrow 1$ and $\chi_+ \rightarrow \sigma$. In terms of the reduced fugacity

$$t \equiv \left(1 - \frac{K}{K_c}\right) \equiv (1 - zK) \tag{33a}$$

we have for $K \rightarrow K_c$

$$x_- = 1 - \frac{z}{z - 2} t \tag{33b}$$

and as a result for $\sigma > 1$ we find that

$$\chi \sim \frac{(\sigma - 1)^2}{2\sigma} \left[\ln\left(\frac{\sigma + 1}{\sigma - 1}\right) \right] \frac{1}{t^2} \tag{34a}$$

whereas for $\sigma = 1$ the result is

$$\chi \sim \frac{1}{2} |\ln t| / t. \tag{34b}$$

The result (31) allows an exact evaluation of the average resistance of walks of length N for a given N . If we expand χ in powers of K , then we have

$$\chi(K) = \sum_N [R]^{(N)} (zK)^N. \tag{35}$$

From (32) the result for one dimension is especially simple:

$$[R]^{(2k-1)} = [R]^{(2k)} = 1 + \frac{1}{3} + \frac{1}{5} + \dots + 1/(2k - 1). \tag{36a}$$

We have verified this simple result by exact enumeration of random walks of length up to 15 steps on a linear lattice. The result does not simply arise by associating each

term in the series of (36a) with a given configuration. Thus we have not seen how to make a simple calculation of this simple-looking result. For large N (36a) becomes

$$[R]^{(N)} \sim \frac{1}{2} \ln N. \tag{36b}$$

For $\sigma > 1$, (34a) implies that for large N

$$[R]^{(N)} \sim \left[\frac{(\sigma - 1)^2}{2\sigma} \ln \left(\frac{\sigma + 1}{\sigma - 1} \right) \right] N \tag{37a}$$

which for large σ becomes

$$[R]^{(N)} \sim \left(1 - \frac{2}{\sigma} \right) N \quad N \gg 1, \sigma \gg 1. \tag{37b}$$

This last result is easily understood as being the simplest approximation which takes account of immediate reversals. In high dimension the walk will essentially be a self-avoiding one. The most important deviation from self-avoidance arises from covering a bond with two successive steps. The probability that this happens is $1/\sigma$ per step. However, such an event deletes two resistances and therefore gives the average resistance per step as $1 - (2/\sigma)$, in agreement with (37b).

3. Conclusion

We have given in (31) and (35) the exact solution for $[R]^{(N)}$, the end-to-end resistance averaged over all N -step random walks on a Cayley tree in which a unit conductance is associated with each step of the walk. For coordination number two, i.e. a one-dimensional lattice, the exact result (36) yields

$$[R]^{(N)} \sim \frac{1}{2} \ln N. \tag{38a}$$

For a Cayley tree with coordination number greater than two we have

$$[R]^{(N)} \sim C_0 N \tag{38b}$$

in agreement with mean-field arguments. Here C_0 is a constant given in (37).

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References

Banavar J R, Harris A B and Koplik J 1983 *Phys. Rev. Lett.* **51** 1115
 Barber M N and Ninham B W 1970 *Random and Restricted Walks* (New York: Gordon and Breach)
 Harris A B and Christou A 1987 to be published
 Harris A B and Lubensky T C 1981 *Phys. Rev. B* **23** 2640
 Koster G F and Slater J C 1954 *Phys. Rev.* **95** 1167
 Montroll E W and Weiss G H 1965 *J. Math. Phys.* **6** 167